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Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type with small data

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ABSTRACT

This paper deals with the quasilinear degenerate Keller–Segel system (KS) of "parabolic–parabolic" type. The global existence of weak solutions to (KS) with small initial data is established when $q \geqslant m + \frac{2}{N}$ (m denotes the intensity of diffusion and q denotes the nonlinearity). In the system of "parabolic–elliptic" type, Sugiyama and Kunii (2006) [13, Theorem 3] and Sugiyama (2007) [12, Theorem 2] state the similar result; note that $q = m + \frac{2}{N}$ corresponds to generalized Fujita's critical exponent. However, the super-critical case where $q \geqslant m + \frac{2}{N}$ has been unsolved for "parabolic–parabolic" type. Therefore this paper gives an answer to the unsolved problem.

1. Introduction and results

The Keller–Segel model was proposed by Keller–Segel [7] in 1970, and is still investigated (see e.g., Kozono and Sugiyama [8], Winkler [17]). The model describes a part of cellular slime molds with the chemotaxis at the life cycle. Usually $u(x,t)$ shows the density of cellular slime molds and $v(x,t)$ shows the density of the semiochemical at place x and time t. Biological and mathematical generalizations of Keller–Segel model are done; note that porous medium-type diffusion is motivated from a biological point of view (see Szymanska, Morales-Rodrigo, Lachowicz and Chaplain [15]) and nonlinear diffusion has been suggested by Hillen and Painter (see their survey [5]).

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Table 1.1
The known results for (KS), $(\mathrm{KS})_0$ and $(\mathrm{KS})_m$.

	(KS)	$(\mathrm{KS})_0$	$(\mathrm{KS})_m$
$q < m + \frac{2}{N}$	Global existence ([13, Theorem 1]: $q \leqslant m$, [6]: $q < m + \frac{2}{N}$)	Global existence [13, Theorem 2]	Global existence ($q = 2$) [2, Theorem 2.4]
$q \geqslant m + \frac{2}{N}$	**Unsolved**	Global existence with small data [13, Theorem 3] Blow-up with large data ($q = 2$) [11, Theorem 1.3]	There exist initial data such that $(\mathrm{KS})_m$ has the blow-up solution [16]

In this paper we deal with the following quasilinear degenerate Keller–Segel system and consider the unsolvable part in our previous paper [6]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot \left(\nabla u^m - u^{q-1}\nabla v\right) & \text{in } \mathbb{R}^N \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{KS}$$

where $N \geqslant 2$, $m \geqslant 1$, $q \geqslant 2$. The initial data (u_0, v_0) satisfies

$$u_0 \geqslant 0, \qquad u_0 \in L^1\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right), \tag{1.1}$$

$$v_0 \geqslant 0, \qquad v_0 \in L^1\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right), \qquad \Delta v_0 \in L^{\frac{N}{2}+1}\left(\mathbb{R}^N\right) \cap L^{\frac{N}{2}(q-m)+1}\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right). \tag{1.2}$$

To confirm current results we introduce the following two problems:

- $(\mathrm{KS})_0$: the parabolic–elliptic Keller–Segel model; the second equation is replaced with

$$0 = \Delta v - v + u$$

(cf. Sugiyama [11], Sugiyama and Kunii [13]).
- $(\mathrm{KS})_m$: the non-degenerate parabolic–parabolic Keller–Segel model with Neumann boundary condition in a ball; u^m is replaced with $(u+\delta)^m$ for some $\delta > 0$ (cf. Cieślak and Winkler [2, Theorem 2.4], Winkler [16]).

In these problems, the studies in Table 1.1 are currently known.

We explain their point of view in [13] including the case where $(\mathrm{KS})_0$. The first equation in (KS) is rewritten as follows:

$$\frac{\partial u}{\partial t} = \Delta u^m - \nabla u^{q-1} \cdot \nabla v - u^{q-1}\Delta v. \tag{E1}$$

In $(\mathrm{KS})_0$, substituting the second equation $\Delta v = v - u$ into (E1) implies

$$\frac{\partial u}{\partial t} = \Delta u^m - \nabla u^{q-1} \cdot \nabla v + u^q - u^{q-1}v; \tag{E2}$$

note that the nonlinear term $-u^{q-1}\Delta v$ in (E1) yields u^q in (E2). Comparing the diffusion term Δu^m with u^q, they derived the L^r- and L^∞-estimate of u and obtained the global solvability of the parabolic–elliptic type $(\mathrm{KS})_0$ when $q < m + \frac{2}{N}$ or when $q \geqslant m + \frac{2}{N}$ and the initial data are sufficiently small. On the other hand, in the parabolic–parabolic type (KS), it is impossible to rewrite (E1) as (E2). In our previous paper [6] we proposed a way to overcome the difficulty, that is, we employed *maximal Sobolev regularity* in parabolic evolution equations (see e.g., Hieber and Prüss [4, Theorem 3.1]):

$$\|\Delta v\|_{L^p(0,T;L^p(\mathbb{R}^N))} \leqslant \|\Delta v_0\|_{L^p(\mathbb{R}^N)} + C_{\langle p\rangle}\|u\|_{L^p(0,T;L^p(\mathbb{R}^N))},$$

where $C_{\langle p\rangle} > 0$ is a constant. This inequality implies that $-u^{q-1}\Delta v$ in (E1) acts like u^q in (E2), and so we could extend the condition $q \leqslant m$ in [13] to $q < m + \frac{2}{N}$ in [6]; note that $q_c := m + \frac{2}{N}$ corresponds to generalized Fujita's exponent. The exponent q_c seems critical because q_c is the border between the global existence and blow-up in the case $(\mathrm{KS})_0$ without restriction on the size of initial data. Moreover, in the case $(\mathrm{KS})_m$ there exist initial data such that the corresponding solution blows up for each $q \geqslant m + \frac{2}{N}$. However, the super-critical case where $q \geqslant m + \frac{2}{N}$ has been unsolved for (KS).

Our purpose in this paper is to establish the global existence of weak solutions to (KS) with small initial data when $q \geqslant m + \frac{2}{N}$. That is, we give an answer to the unsolved part in Table 1.1.

Before stating our result we define global weak solutions to (KS).

Definition 1.1. Let $T > 0$. A pair (u, v) of non-negative functions defined on $\mathbb{R}^N \times (0, T)$ is called a *weak solution* to (KS) *on* $[0, T)$ if

(a) $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ $(\forall p \in [1, \infty])$, $u^m \in L^2(0, T; H^1(\mathbb{R}^N))$,
(b) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$,
(c) (u, v) satisfies (KS) in the distributional sense, i.e., for every $\varphi \in C_0^\infty(\mathbb{R}^N \times [0, T))$,

$$\int_0^T \int_{\mathbb{R}^N} \left(\nabla u^m \cdot \nabla\varphi - u^{q-1}\nabla v \cdot \nabla\varphi - u\varphi_t\right) dx\,dt = \int_{\mathbb{R}^N} u_0(x)\varphi(x, 0)\,dx,$$

$$\int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla\varphi + v\varphi - u\varphi - v\varphi_t)\,dx\,dt = \int_{\mathbb{R}^N} v_0(x)\varphi(x, 0)\,dx.$$

In particular, if $T > 0$ can be taken arbitrary, then (u, v) is called a *global weak solution* to (KS).

We now state our main result in this paper.

Theorem 1.1. *Let $N \geqslant 2$, $m \geqslant 1$, $q \geqslant 2$, $T > 0$. Let m and q satisfy*

$$q \geqslant m + \frac{2}{N}$$

and let (u_0, v_0) satisfy (1.1), (1.2). *Assume further that $\|u_0\|_{L^{\frac{N}{2}}}$, $\|u_0\|_{L^{\frac{N}{2}(q-m)}}$, $\|\Delta v_0\|_{L^{\frac{N}{2}+1}}$, $\|\Delta v_0\|_{L^{\frac{N}{2}+q-1}}$, $\|\Delta v_0\|_{L^{\frac{N}{2}(q-m)+1}}$ and $\|\Delta v_0\|_{L^{\frac{N}{2}(q-m)+q-1}}$ satisfy the following smallness conditions*:

$$\|u_0\|_{L^{q_0}}, \|u_0\|_{L^{q_*}} \leqslant \min\{\delta_{u,q_0}, \delta_{u,q_*}, \delta_{u,r_2}\},$$

$$\frac{1}{q_0}\|\Delta v_0\|_{L^{q_0+1}}^{q_0+1} + \frac{1}{q_0+q-1}\|\Delta v_0\|_{L^{q_0+q-1}}^{q_0+q-1} < \min\{\delta_{v,q_0+1}, \delta_{v,q_*+1}, \delta_{v,r_2+1}\},$$

$$\frac{1}{q_*}\|\Delta v_0\|_{L^{q_*+1}}^{q_*+1} + \frac{1}{q_*+q-1}\|\Delta v_0\|_{L^{q_*+q-1}}^{q_*+q-1} < \min\{\delta_{v,q_0+1}, \delta_{v,q_*+1}, \delta_{v,r_2+1}\},$$

where $q_0 = \frac{N}{2}$, $q_* = \frac{N}{2}(q-m)$, $r_2 = \max\{3, q_0, 2q_* - m + 1, N - m + 1, q_* + q - 1\}$, $\delta_{u,r} = \delta_{u,r}(r, m, q, N)$ *and* $\delta_{v,r} = \delta_{v,r}(r, m, q, N)$ *are positive constants defined as in* (4.6) *and* (4.7) *in Section* 4*. Then there exists a non-negative* (*global*) *weak solution* (u, v) *to* (KS) *on* $[0, T)$*. Moreover,* $u^m \in C((0, T); L^p_{\mathrm{loc}}(\mathbb{R}^N))$ $(\forall p \in [1, \infty))$ *and the following estimates hold*:

$$\|u\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} + \|v\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} \leqslant K_1 \quad \left(\forall r \in \left[\frac{N}{2}(q-m)+1, \infty\right]\right), \tag{1.3}$$

$$\|v_t\|_{L^r(0,T;L^r(\mathbb{R}^N))} + \|v\|_{L^r(0,T;W^{2,r}(\mathbb{R}^N))} \leqslant K_2 \quad \left(\forall r \in \left[\frac{N}{2}(q-m)+1, \infty\right)\right), \tag{1.4}$$

where $K_1 = K_1(r, \|u_0\|_{L^r}, \|v_0\|_{L^r}, \|\Delta v_0\|_{L^r}, m, q, N, T) > 0$ *and* $K_2 = K_2(K_1, T) > 0$ *are constants.*

The key to the proof of Theorem 1.1 lies in the L^r- and L^∞-estimate of u. Using maximal Sobolev regularity as stated above, we can obtain the L^r-estimate of u by assuming that

$$\|\Delta v_0\|^{\frac{N}{2}+q-1}_{L^{\frac{N}{2}+q-1}(\mathbb{R}^N)}, \|\Delta v_0\|^{\frac{N}{2}(q-m)+q-1}_{L^{\frac{N}{2}(q-m)+q-1}(\mathbb{R}^N)} \leqslant Mr^{-\frac{N}{2}},$$

where $M = M(m, q, N) > 0$ is a constant. In order to obtain the L^∞-estimate of u, if $r \to \infty$, then it should be $\Delta v_0 = 0$. For this reason, we try to derive the L^∞-estimate based on *the method of* L^∞-L^r-*estimate* in Suzuki [14]. His method is developed for the following equation:

$$\frac{\partial u}{\partial t} = \Delta u^m + a\nabla u^p + u^q \quad \text{in } \mathbb{R}^N \times (0, T),$$

where $a \neq 0$. Noting that this equation is similar to (E2) and referring to his method, once we have the L^{r_*}-estimate for some r_*, we obtain the L^∞-estimate of u. Thus we construct a global weak solution to (KS) with small initial data when $q \geqslant m + \frac{2}{N}$.

In [17], they studied (KS) with $m = 1$, $q = 2$ and homogeneous Neumann boundary conditions in a smooth bounded domain. They established the following assertion. If the initial data $(u_0, \nabla v_0)$ is small, then the solution is global in time and bounded and asymptotically behaves like the solution of a discoupled system of linear parabolic equations (see also Corrias and Perthame [3]). On the other hand, because of the use of our key tool "maximal Sobolev regularity", our assertion needs the smallness conditions involving Δv_0 and does not give the boundedness and asymptotic behavior of the solution. So we would like to replace the smallness conditions for the initial data by appropriate hypotheses and study the behavior of the solution more precisely in our future work. Moreover, in order to somehow complete the lower left (unsolved) box in Table 1.1, we will discuss the case where $q \geqslant m + \frac{2}{N}$ with large initial data in our forthcoming paper.

This paper is organized as follows. In Section 2 we introduce some basic inequalities. In Section 3 we introduce an approximate problem and explain how to construct global approximate solutions. Section 4 gives the proof of L^r-estimates of approximate solutions. Section 5 presents the proof of L^∞-estimates of approximate solutions, which is the main part of this paper. Finally we prove Theorem 1.1 in Section 6.

2. Basic inequalities

We start with the fundamental estimates of solutions to the following Cauchy problem for inhomogeneous linear heat equations:

$$\begin{cases} \dfrac{\partial z}{\partial t} = \Delta z - z + f & \text{in } \mathbb{R}^N \times (0, T), \\ z(x, 0) = z_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{LH}$$

The following lemma is obtained by combining the semigroup theory with L^p-L^q-estimates for the heat semigroup (see [13, Lemma 5]) and by using a particular consequence of well-known results on maximal Sobolev regularity in parabolic evolution equations (see [4, Theorem 3.1], Ladyženskaja, Solonnikov and Ural'ceva [9, Section 3, Chapter IV]). The constant $C_{\langle p\rangle}$ in (2.5) originates in Stein [10, Theorem 3, Chapter IV].

Lemma 2.1. *Let $N \in \mathbb{N}$, $T > 0$, $1 \leqslant p \leqslant \infty$ and $z_0 \in L^p(\mathbb{R}^N)$. If $f \in L^1(0, T; L^p(\mathbb{R}^N))$, then* (LH) *has a unique mild solution $z \in C([0, T]; L^p(\mathbb{R}^N))$ given by*

$$z(t) = e^{-t} e^{t\Delta} z_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} f(s)\, ds, \quad t \in [0, T],$$

where $(e^{t\Delta} f)(x, t) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y, t)\, dy$. Moreover, the following estimates for the solution z hold.

(I) L^p-L^q-estimates. *Assume that $1 \leqslant q \leqslant p \leqslant \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{N}$, $z_0 \in W^{2,p}(\mathbb{R}^N)$ and $f \in L^\infty(0, T; W^{1,q}(\mathbb{R}^N))$. Then for every $t \in [0, T]$,*

$$\left\| z(t) \right\|_{L^p(\mathbb{R}^N)} \leqslant e^{-t} \|z_0\|_{L^p(\mathbb{R}^N)} + C_0 \|f\|_{L^\infty(0,T;L^q(\mathbb{R}^N))}, \tag{2.1}$$

$$\left\| \nabla z(t) \right\|_{L^p(\mathbb{R}^N)} \leqslant e^{-t} \|\nabla z_0\|_{L^p(\mathbb{R}^N)} + C_0 \|f\|_{L^\infty(0,T;L^q(\mathbb{R}^N))}, \tag{2.2}$$

$$\left\| \Delta z(t) \right\|_{L^p(\mathbb{R}^N)} \leqslant e^{-t} \|\Delta z_0\|_{L^p(\mathbb{R}^N)} + C_0 \|\nabla f\|_{L^\infty(0,T;L^q(\mathbb{R}^N))}, \tag{2.3}$$

where C_0 is a positive constant depending on p, q and N.

(II) Maximal Sobolev regularity. *Assume that $1 < p < \infty$ and $f \in L^p(0, T; L^p(\mathbb{R}^N))$. Then for every $t \in [0, T]$,*

$$\|\Delta z\|_{L^p(0,t;L^p(\mathbb{R}^N))} \leqslant \|\Delta z_0\|_{L^p(\mathbb{R}^N)} \left(1 - e^{-pt}\right)^{\frac{1}{p}} + C_{\langle p\rangle} \|f\|_{L^p(0,t;L^p(\mathbb{R}^N))}, \tag{2.4}$$

where $C_{\langle p\rangle} = C_{\langle p\rangle}(p, N) > 0$ is a constant. In particular, when $p > 2$, $C_{\langle p\rangle}$ is given by

$$C_{\langle p\rangle} = A_0 \left(A_1 p + A_2 \frac{p}{p-2} \right)^{\frac{2(p-1)}{p}} \left(\frac{5^N p}{2} \right)^{\frac{p-2}{2p}}, \tag{2.5}$$

where $A_0 = A_0(N)$, $A_1 = A_1(N)$ and $A_2 = A_2(N)$ are positive constants.

The following lemma is given by the Gagliardo–Nirenberg type inequality, which is proved by [11, Lemma 2.4].

Lemma 2.2. *Let $N \in \mathbb{N}$, $m \geqslant 1$, $a > 2$, $u \in L^{q_1}(\mathbb{R}^N)$ with $q_1 \geqslant 1$ and $u^{\frac{r+m-1}{2}} \in H^1(\mathbb{R}^N)$ with $r > 2$. If $q_1 \in [1, r+m-1]$ and $q_2 \in [\frac{r+m-1}{2}, \frac{a(r+m-1)}{2}]$ satisfy*

$$\begin{cases} 1 \leqslant q_1 \leqslant q_2 \leqslant \infty & (N = 1), \\ 1 \leqslant q_1 \leqslant q_2 < \infty & (N = 2), \\ 1 \leqslant q_1 \leqslant q_2 \leqslant \dfrac{N(r+m-1)}{N-2} & (N \geqslant 3), \end{cases}$$

then

$$\|u\|_{L^{q_2}(\mathbb{R}^N)} \leqslant c(N)^{\frac{2}{r+m-1}} \|u\|_{L^{q_1}(\mathbb{R}^N)}^{1-\theta} \left\| \nabla u^{\frac{r+m-1}{2}} \right\|_{L^2(\mathbb{R}^N)}^{\frac{2\theta}{r+m-1}},$$

where

$$\theta = \frac{r+m-1}{2}\left(\frac{1}{q_1}-\frac{1}{q_2}\right)\left(\frac{1}{N}-\frac{1}{2}+\frac{r+m-1}{2q_1}\right)^{-1},$$

$$c(N) = \begin{cases} c(N,a) & (\frac{r+m-1}{2} \leqslant q_1), \\ c_0(N,a)^{\frac{1}{\beta}} & (1 \leqslant q_1 < \frac{r+m-1}{2}), \end{cases}$$

$$\beta = \frac{q_2 - \frac{r+m-1}{2}}{q_2 - q_1}\left[\frac{2q_1}{r+m-1} + \left(1 - \frac{2q_1}{r+m-1}\right)\frac{2N}{N+2}\right].$$

The following lemma is derived from the usual Gagliardo–Nirenberg inequality and the Young inequality (see [14, Lemma 2.9]).

Lemma 2.3. *Let $N \in \mathbb{N}$, $m \geqslant 1$, $q \geqslant 2$, $r \geqslant m+q-2$ and $I = [a,b] \subset \mathbb{R}$. Let $\alpha = \frac{2(r-q+1)}{r+m-q}$, $\tilde{\alpha} = 2(\frac{\alpha}{N}+1)$, $k = 1+\frac{2}{N}$ and $f \in C(I; L^\alpha(\mathbb{R}^N)) \cap L^2(I; H^1(\mathbb{R}^N))$. Then*

$$\left[\int_I \int_{\mathbb{R}^N} |f|^{\tilde{\alpha}}\, dx\, dt\right]^{\frac{1}{k}} \leqslant C(N)^{\frac{1}{k}}\left[\max_{t \in I} \int_{\mathbb{R}^N} |f|^\alpha\, dx + \int_I \int_{\mathbb{R}^N} |\nabla f|^2\, dx\, dt\right],$$

where $C(N)$ is a positive constant depending only on N.

3. Construction of approximate solutions

We consider the following approximate problem:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \nabla \cdot \left(\nabla (u_\varepsilon + \varepsilon)^m - \left(u_\varepsilon + \varepsilon^{\frac{m-1}{q-2}}\right)^{q-2} u_\varepsilon \nabla v_\varepsilon\right) & \text{in } \mathbb{R}^N \times (0,T), \qquad \ldots(1)_\varepsilon \\ \dfrac{\partial v_\varepsilon}{\partial t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon & \text{in } \mathbb{R}^N \times (0,T), \qquad \ldots(2)_\varepsilon \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad v_\varepsilon(x,0) = v_{0\varepsilon}(x), & x \in \mathbb{R}^N, \end{cases} \qquad (\mathrm{KS})_\varepsilon$$

where $N \in \mathbb{N}$, $m \geqslant 1$, $q \geqslant 2$ and $\varepsilon \in (0,1)$; note that the exponent of ε is different from [13] and [6]. The initial data $u_{0\varepsilon}, v_{0\varepsilon} \in C_0^\infty(\mathbb{R}^N)$ are given as $u_{0\varepsilon} := (\rho_\varepsilon * u_0)\zeta_\varepsilon$, $v_{0\varepsilon} := (\rho_\varepsilon * v_0)\zeta_\varepsilon$, where ρ_ε is a mollifier such that

$$0 \leqslant \rho_\varepsilon \in C_0^\infty\left(\mathbb{R}^N\right), \quad \operatorname{supp} \rho_\varepsilon \subset \overline{B(0,\varepsilon)}, \qquad \int_{\mathbb{R}^N} \rho_\varepsilon(x)\, dx = 1,$$

and ζ_ε is a cut-off function, i.e., $\zeta_\varepsilon(x) := \zeta(\varepsilon x)$, where ζ is a fixed function in $C_0^\infty(\mathbb{R}^N)$ such that

$$0 \leqslant \zeta \leqslant 1, \quad \zeta(x) = 1 \quad \left(|x| \leqslant 1\right), \qquad \zeta(x) = 0 \quad \left(|x| \geqslant 2\right).$$

First we state the existence result on local solutions to $(\mathrm{KS})_\varepsilon$.

Proposition 3.1 *(Local existence of approximate solutions). Let $N \in \mathbb{N}$, $m \geqslant 1$, $q \geqslant 2$, and $\varepsilon \in (0,1)$. Then there exists $T_1 = T_1(\varepsilon, \|u_{0\varepsilon}\|_{W^{2,N+2}}, \|v_{0\varepsilon}\|_{W^{1,\infty}}, \|\Delta v_{0\varepsilon}\|_{L^\infty}, m, q, N) > 0$ such that $(\mathrm{KS})_\varepsilon$ has a unique non-negative solution $(u_\varepsilon, v_\varepsilon)$ on $[0, T_1)$ such that*

$$u_\varepsilon \in W^{1,N+2}\big(0, T_1; L^{N+2}\big(\mathbb{R}^N\big)\big) \cap L^\infty\big(0, T_1; W^{2,N+2}\big(\mathbb{R}^N\big)\big),$$
$$v_\varepsilon \in C^1\big([0, T_1); L^{N+2}\big(\mathbb{R}^N\big)\big) \cap C\big([0, T_1); W^{2,N+2}\big(\mathbb{R}^N\big)\big) \cap L^\infty\big(0, T_1; L^\infty\big(\mathbb{R}^N\big)\big).$$

Moreover, u_ε has the mass conservation law:

$$\big\|u_\varepsilon(t)\big\|_{L^1(\mathbb{R}^N)} = \|u_{0\varepsilon}\|_{L^1(\mathbb{R}^N)}, \quad t \in [0, T_1). \tag{3.1}$$

This proposition is essentially proved in [13, Proposition 8, Lemmas 11 and 12] by linearizing the first equation $(1)_\varepsilon$ and applying the analytic semigroup theory (Amann [1, Theorem IV.1.5.1]) and using the contraction mapping principle.

Remark 3.1. Let $T > 0$. Let u_ε be a local solution to (KS) on $[0, T)$ in Proposition 3.1. Then the following continuity holds:

$$\big\|u_\varepsilon(t)\big\|_{L^r(\mathbb{R}^N)} \in C\big([0, T]\big) \quad \big(\forall r \in [1, \infty)\big). \tag{3.2}$$

Indeed, reading the standard argument to construct the local (approximate) solution again, we see that for every $\alpha \in (N, \infty)$,

$$u_\varepsilon \in W^{1,\alpha}\big(0, T; L^\alpha\big(\mathbb{R}^N\big)\big) \cap L^\infty\big(0, T; W^{2,\alpha}\big(\mathbb{R}^N\big)\big).$$

In particular,

$$u_\varepsilon \in C\big([0, T]; L^\alpha\big(\mathbb{R}^N\big)\big).$$

This fact together with the mass conservation law (3.1) implies the continuity (3.2). This continuity will be used in Section 4.

Next we state two key propositions for L^r- and L^∞-estimates of approximate solutions. Their proofs will be given in the next two sections.

Proposition 3.2 *(L^r-estimate ($1 \leqslant r \leqslant r_2$)). Let $N \geqslant 2$, $m \geqslant 1$, $q \geqslant 2$, $\varepsilon \in (0,1)$ and $T > 0$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(\mathrm{KS})_\varepsilon$ on $[0, T)$. Assume further that*

$$q \geqslant m + \frac{2}{N} \tag{3.3}$$

and (u_0, v_0) satisfies the smallness conditions as in Theorem 1.1. Then the following estimate holds:

$$\sup_{0<t<T} \big\|u_\varepsilon(t)\big\|_{L^r(\mathbb{R}^N)} \leqslant K_{\varepsilon,r} \quad \big(\forall r \in [1, r_2]\big), \tag{3.4}$$

where $K_{\varepsilon,r} = K_{\varepsilon,r}(\varepsilon, r, \|u_{0\varepsilon}\|_{L^r}, \|\Delta v_{0\varepsilon}\|_{L^{r+1}}, \|\Delta v_{0\varepsilon}\|_{L^{r+q-1}}, m, q)$ $(r_1 \leqslant r \leqslant r_2)$, $K_{\varepsilon,r} = \|u_{0\varepsilon}\|_{L^1} + K_{\varepsilon,r_1}$ $(1 \leqslant r < r_1)$, $r_1 = r_1(m, q, N)$ and $r_2 = r_2(m, q, N)$ are positive constants.

Proposition 3.3 *(L^∞-estimate). Let $N \geqslant 2$, $m \geqslant 1$, $q \geqslant 2$, $\varepsilon \in (0,1)$ and $T > 0$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *on* $[0,T)$*. Assume further that m and q satisfy* (3.3) *and (u_0, v_0) satisfies the smallness conditions as in Theorem* 1.1*. Then the following estimate holds*:

$$\|u_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^N))} \leqslant K_{\varepsilon,\infty}, \tag{3.5}$$

where $K_{\varepsilon,\infty} > 0$ is a constant.

Once we obtain the L^∞-estimate of the first component u_ε of the solution $(u_\varepsilon, v_\varepsilon)$ to the approximate problem, the $W^{2,N+2}$-estimate follows from [13, Lemma 13].

Proposition 3.4. *Let $N \in \mathbb{N}$, $m \geqslant 1$, $q \geqslant 2$, $\varepsilon \in (0,1)$ and $T > 0$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *on* $[0,T)$*. Assume that*

$$\|u_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^N))} \leqslant K_\varepsilon < \infty.$$

Then there exists a constant $M_\varepsilon = M_\varepsilon(\varepsilon, u_{0\varepsilon}, v_{0\varepsilon}, K_\varepsilon, m, q, N, T) > 0$ such that

$$\|u_\varepsilon\|_{L^\infty(0,T;W^{2,N+2}(\mathbb{R}^N))} \leqslant M_\varepsilon.$$

Combining Propositions 3.3 and 3.4, we can construct a global solution to $(\mathrm{KS})_\varepsilon$.

Proposition 3.5 *(Global existence of approximate solutions). Let $N \geqslant 2$, $m \geqslant 1$ and $q \geqslant 2$. Assume further that m and q satisfy* (3.3) *and (u_0, v_0) satisfies the smallness conditions as in Theorem* 1.1*. Then* $(\mathrm{KS})_\varepsilon$ *has a unique non-negative global solution $(u_\varepsilon, v_\varepsilon)$ such that*

$$u_\varepsilon \in W^{1,N+2}\big(0,T;L^{N+2}\big(\mathbb{R}^N\big)\big) \cap L^\infty\big(0,T;W^{2,N+2}\big(\mathbb{R}^N\big)\big) \quad (\forall T > 0),$$
$$v_\varepsilon \in C^1\big([0,T);L^{N+2}\big(\mathbb{R}^N\big)\big) \cap C\big([0,T);W^{2,N+2}\big(\mathbb{R}^N\big)\big) \cap L^\infty\big(0,T;L^\infty\big(\mathbb{R}^N\big)\big) \quad (\forall T > 0).$$

Proof. Let $(u_\varepsilon, v_\varepsilon)$ be the unique non-negative solution to $(\mathrm{KS})_\varepsilon$ on $[0,T_1)$ obtained by Proposition 3.1. From (3.5) we can apply Proposition 3.4, and so we have

$$\|u_\varepsilon\|_{L^\infty(0,T;W^{2,N+2}(\mathbb{R}^N))} \leqslant M_\varepsilon, \tag{3.6}$$

where $M_\varepsilon = M_\varepsilon(\varepsilon, u_{0\varepsilon}, v_{0\varepsilon}, K_{\varepsilon,\infty}, m, q, N, T_1) > 0$ is a constant. Applying (2.1), (2.2) and (2.3) with $z = v_\varepsilon$ and $f = u_\varepsilon$, we see from (3.6) that

$$\sup_{0<t<T_1} \big\|v_\varepsilon(t)\big\|_{W^{1,\infty}(\mathbb{R}^N)} \leqslant M'_\varepsilon, \qquad \sup_{0<t<T_1} \big\|\Delta v_\varepsilon(t)\big\|_{L^\infty(\mathbb{R}^N)} \leqslant M'_\varepsilon,$$

where $M'_\varepsilon > 0$ is a constant. Therefore it follows that $(u_\varepsilon, v_\varepsilon)$ can be extended as a solution to $(\mathrm{KS})_\varepsilon$ on $[0, T_1 + T_2)$ for some $T_2 > 0$. Repeating this argument, we conclude that $(\mathrm{KS})_\varepsilon$ has a unique global solution. □

4. L^r-estimates

In this section we prove the L^r-estimate (3.4) of approximate solutions. We consider estimates of the diffusion (good) term (see $\boldsymbol{I}_1$ below) and the nonlinear (bad) term (see $\boldsymbol{I}_2$ below) like those in [6, Section 4].

Proof of Proposition 3.2. Let $r \in (1, \infty)$. Multiplying the first approximate equation $(1)_\varepsilon$ (see Section 3) by u_ε^{r-1} and integrating it over $\mathbb{R}^N$, we obtain

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r &= -\int_{\mathbb{R}^N} \nabla(u_\varepsilon+\varepsilon)^m \cdot \nabla u_\varepsilon^{r-1}\,dx + \int_{\mathbb{R}^N}\left(\left(u_\varepsilon+\varepsilon^{\frac{m-1}{q-2}}\right)^{q-2} u_\varepsilon \nabla v_\varepsilon\right)\cdot \nabla u_\varepsilon^{r-1}\,dx \\ &=: -\boldsymbol{I}_1 + \boldsymbol{I}_2. \end{aligned} \tag{4.1}$$

First we deal with $\boldsymbol{I}_1$:

$$\begin{aligned} -\boldsymbol{I}_1 &= -m(r-1)\int_{\mathbb{R}^N} (u_\varepsilon+\varepsilon)^{m-1}\nabla u_\varepsilon \cdot u_\varepsilon^{r-2}\nabla u_\varepsilon\,dx \\ &\leqslant -m(r-1)\int_{\mathbb{R}^N}\left|u_\varepsilon^{\frac{r+m-3}{2}}\nabla u_\varepsilon\right|^2 dx - m(r-1)\varepsilon^{m-1}\int_{\mathbb{R}^N}\left|u_\varepsilon^{\frac{r-2}{2}}\nabla u_\varepsilon\right|^2 dx \\ &= -\frac{4m(r-1)}{(r+m-1)^2}\left\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\right\|_{L^2(\mathbb{R}^N)}^2 - \frac{4m(r-1)\varepsilon^{m-1}}{r^2}\left\|\nabla u_\varepsilon^{\frac{r}{2}}(t)\right\|_{L^2(\mathbb{R}^N)}^2 \\ &=: -\boldsymbol{I}_1' - \boldsymbol{I}_1''. \end{aligned} \tag{4.2}$$

Next we consider the estimate of $\boldsymbol{I}_2$ (refer to [6, Section 4]). Setting

$$F(s) := \int_0^s \left(\tau + \varepsilon^{\frac{m-1}{q-2}}\right)^{q-2}\tau^{r-1}\,d\tau, \quad s \geqslant 0,\ \varepsilon \in (0,1)$$

and noting that $F(s) \leqslant 2^{q-2}[s^{r+q-2}/(r+q-2) + \varepsilon^{m-1}s^r/r]$ $(s \geqslant 0,\ \varepsilon \in (0,1))$, we have

$$\boldsymbol{I}_2 \leqslant \frac{2^{q-2}(r-1)}{r+q-2}\int_{\mathbb{R}^N} u_\varepsilon^{r+q-2}|\Delta v_\varepsilon|\,dx + \frac{2^{q-2}\varepsilon^{m-1}(r-1)}{r}\int_{\mathbb{R}^N} u_\varepsilon^r|\Delta v_\varepsilon|\,dx =: \boldsymbol{I}_2' + \boldsymbol{I}_2''.$$

Using Hölder's inequality, maximal Sobolev regularity (2.4) and Young's inequality with the (mutually dual) exponents $r+q-1$ ($\geqslant r+1 > 1$ due to $q \geqslant 2$) and $\frac{r+q-1}{r+q-2}$, we see that for $r \geqslant \max\{\frac{N}{2}, \frac{N}{2}(q-m)\}$,

$$\begin{aligned}\frac{r+q-2}{2^{q-2}(r-1)}\int_0^t \boldsymbol{I}_2'\,ds &= \int_0^t\int_{\mathbb{R}^N} u_\varepsilon^{r+q-2}|\Delta v_\varepsilon|\,dx\,ds \\ &\leqslant \left(\int_0^t\int_{\mathbb{R}^N} u_\varepsilon^{r+q-1}\,dx\,ds\right)^{\frac{r+q-2}{r+q-1}}\left(\int_0^t\int_{\mathbb{R}^N}|\Delta v_\varepsilon|^{r+q-1}\,dx\,ds\right)^{\frac{1}{r+q-1}}\end{aligned}$$

$$\leqslant \|\Delta v_{0\varepsilon}\|_{L^{r+q-1}(\mathbb{R}^N)}\bigl(1-e^{-(r+q-1)t}\bigr)^{\frac{1}{r+q-1}}\left(\int_0^t\int_{\mathbb{R}^N} u_\varepsilon^{r+q-1}\,dx\,ds\right)^{\frac{r+q-2}{r+q-1}}$$

$$+\,C_{\langle r+q-1\rangle}\int_0^t\int_{\mathbb{R}^N} u_\varepsilon^{r+q-1}\,dx\,ds$$

$$\leqslant \|\Delta v_{0\varepsilon}\|_{L^{r+q-1}}^{r+q-1}\bigl(1-e^{-(r+q-1)t}\bigr)+(1+C_{\langle r+q-1\rangle})\int_0^t\int_{\mathbb{R}^N} u_\varepsilon^{r+q-1}\,dx\,ds. \tag{4.3}$$

Now let $r \geqslant r_0 := \max\{\frac{N}{2}, \frac{N}{2}(q-m), N(q-m)-m+1\}$. Noting that

$$q \geqslant m+\frac{2}{N} \quad \Leftrightarrow \quad 1 \leqslant \frac{N}{2}(q-m),$$

we can apply Lemma 2.2 with $q_1 = \frac{N}{2}(q-m)$, $q_2 = r+q-1$ and

$$a = \begin{cases} 3 & (\text{when } N = 1, 2),\\ \frac{2N}{N-2} & (\text{when } N \geqslant 3).\end{cases}$$

Therefore it follows from Lemma 2.2 that

$$\int_0^t \boldsymbol{I}_2'\,ds \leqslant \frac{2^{q-2}(r-1)}{r+q-2}\Biggl[\|\Delta v_{0\varepsilon}\|_{L^{r+q-1}(\mathbb{R}^N)}^{r+q-1}\bigl(1-e^{-(r+q-1)t}\bigr)$$

$$+\,C_2'\int_0^t \bigl\|u_\varepsilon(s)\bigr\|_{L^{\frac{N}{2}(q-m)}(\mathbb{R}^N)}^{q-m}\bigl\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(s)\bigr\|_{L^2(\mathbb{R}^N)}^2\,ds\Biggr], \tag{4.4}$$

where $C_2' = C_2'(c(N), r, m, q, N) > 0$ is a constant. Let $N \geqslant 2$ and $r \geqslant r_1 := \max\{r_0, N-m+1, m-3\}$. The same argument as above to $\boldsymbol{I}_2''$ gives

$$\int_0^t \boldsymbol{I}_2''\,ds \leqslant \frac{2^{q-2}\varepsilon^{m-1}(r-1)}{r}\Biggl[\|\Delta v_{0\varepsilon}\|_{L^{r+1}(\mathbb{R}^N)}^{r+1}\bigl(1-e^{-(r+1)t}\bigr)$$

$$+\,C_2''\int_0^t \bigl\|u_\varepsilon(s)\bigr\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}\bigl\|\nabla u_\varepsilon^{\frac{r}{2}}(s)\bigr\|_{L^2(\mathbb{R}^N)}^2\,ds\Biggr], \tag{4.5}$$

where $C_2'' = C_2''(c(N), r, N) > 0$ is a constant. Now we set

$$q_0 := \frac{N}{2}, \qquad q_* := \frac{N}{2}(q-m),$$

$$\delta_{u,r} := \min\left\{1, \frac{m}{2^{q-2}rC_2''}, \left(\frac{m(r+q-2)}{2^{q-2}(r+m-1)^2C_2'}\right)^{\frac{1}{q-m}}\right\}, \tag{4.6}$$

$$\delta_{v,r+1} := \min\left\{1, \left(\frac{m}{2^{q-2}C_2''r}\right)^{q_0}\left(2^{q-2}q_0(q_0-1)\right)^{-1}, \left(\frac{m(r+q-2)}{2^{q-2}C_2'(r+m-1)^2}\right)^{q_0}\left(2^{q-2}q_*(q_*-1)\right)^{-1}\right\}. \tag{4.7}$$

Integrating (4.1) over $(0,t)$, we see from (4.2), (4.4) and (4.5) that

$$\begin{aligned}\left\|u_\varepsilon(t)\right\|_{L^r(\mathbb{R}^N)}^r \leqslant{}& \|u_{0\varepsilon}\|_{L^r}^r \\ &+ \varepsilon^{m-1}\int_0^t\left[-\frac{4m(r-1)}{r} + 2^{q-2}(r-1)C_2''\left\|u_\varepsilon(s)\right\|_{L^{q_0}(\mathbb{R}^N)}\right]\left\|\nabla u_\varepsilon^{\frac{r}{2}}(s)\right\|_{L^2(\mathbb{R}^N)}^2 ds \\ &+ \int_0^t\left[-\frac{4mr(r-1)}{(r+m-1)^2} + \frac{2^{q-2}r(r-1)C_2'}{r+q-2}\left\|u_\varepsilon(s)\right\|_{L^{q_*}(\mathbb{R}^N)}^{q-m}\right]\left\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(s)\right\|_{L^2(\mathbb{R}^N)}^2 ds \\ &+ 2^{q-2}r(r-1)\left(\frac{\varepsilon^{m-1}}{r}\|\Delta v_{0\varepsilon}\|_{L^{r+1}}^{r+1} + \frac{1}{r+q-2}\|\Delta v_{0\varepsilon}\|_{L^{r+q-1}}^{r+q-1}\right).\end{aligned} \tag{4.8}$$

Note that $\|u_\varepsilon(s)\|_{L^{q_0}(\mathbb{R}^N)}, \|u_\varepsilon(s)\|_{L^{q_*}(\mathbb{R}^N)} \in C([0,T])$ (see Remark 3.1). From this continuity, if $\|u_{0\varepsilon}\|_{L^{q_0}} \leqslant \delta_{u,q_0}$ and $\|u_{0\varepsilon}\|_{L^{q_*}} \leqslant \delta_{u,q_0}$, then we can find a time t_1 from (4.8) with $r = q_0$ such that for $t \in (0,t_1)$,

$$\left\|u_\varepsilon(t)\right\|_{L^{q_0}(\mathbb{R}^N)}^{q_0} \leqslant \|u_{0\varepsilon}\|_{L^{q_0}}^{q_0} + 2^{q-2}r(r-1)\left(\frac{\varepsilon^{m-1}}{r}\|\Delta v_{0\varepsilon}\|_{L^{r+1}}^{r+1} + \frac{1}{r+q-2}\|\Delta v_{0\varepsilon}\|_{L^{r+q-1}}^{r+q-1}\right)\bigg|_{r=q_0}. \tag{4.9}$$

If $\|u_{0\varepsilon}\|_{L^{q_0}} \leqslant \delta_{u,q_*}$ and $\|u_{0\varepsilon}\|_{L^{q_*}} \leqslant \delta_{u,q_*}$, then there exists a time t_2 from (4.8) with $r = q_*$ such that for $t \in (0,t_2)$,

$$\left\|u_\varepsilon(t)\right\|_{L^{q_*}(\mathbb{R}^N)}^{q_*} \leqslant \|u_{0\varepsilon}\|_{L^{q_*}}^{q_*} + 2^{q-2}r(r-1)\left(\frac{\varepsilon^{m-1}}{r}\|\Delta v_{0\varepsilon}\|_{L^{r+1}}^{r+1} + \frac{1}{r+q-2}\|\Delta v_{0\varepsilon}\|_{L^{r+q-1}}^{r+q-1}\right)\bigg|_{r=q_*}. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8) and then setting $r = q_0$ and $r = q_*$, we see that if

$$\begin{aligned}&\frac{1}{q_0}\|\Delta v_{0\varepsilon}\|_{L^{q_0+1}}^{q_0+1} + \frac{1}{q_0+q-1}\|\Delta v_{0\varepsilon}\|_{L^{q_0+q-1}}^{q_0+q-1} \leqslant \min\{\delta_{v,q_0+1}, \delta_{v,q_*+1}\}, \\ &\frac{1}{q_*}\|\Delta v_{0\varepsilon}\|_{L^{q_*+1}}^{q_*+1} + \frac{1}{q_*+q-1}\|\Delta v_{0\varepsilon}\|_{L^{q_*+q-1}}^{q_*+q-1} \leqslant \min\{\delta_{v,q_0+1}, \delta_{v,q_*+1}\},\end{aligned}$$

then (4.9) and (4.10) hold for all $t \in (0, 2\min\{t_1,t_2\})$. Repeating this argument, we obtain (4.9) and (4.10) for $t \in (0,T)$. Assume that $u_{0\varepsilon}$ and $v_{0\varepsilon}$ satisfy

$$\|u_{0\varepsilon}\|_{L^{q_0}},\ \|u_{0\varepsilon}\|_{L^{q_*}} \leqslant \min\{\delta_{u,q_0}, \delta_{u,q_*}, \delta_{u,r_2}\},$$
$$\frac{1}{q_0}\|\Delta v_{0\varepsilon}\|_{L^{q_0+1}}^{q_0+1} + \frac{1}{q_0+q-1}\|\Delta v_{0\varepsilon}\|_{L^{q_0+q-1}}^{q_0+q-1} \leqslant \min\{\delta_{v,q_0+1}, \delta_{v,q_*+1}, \delta_{v,r_2+1}\},$$
$$\frac{1}{q_*}\|\Delta v_{0\varepsilon}\|_{L^{q_*+1}}^{q_*+1} + \frac{1}{q_*+q-1}\|\Delta v_{0\varepsilon}\|_{L^{q_*+q-1}}^{q_*+q-1} \leqslant \min\{\delta_{v,q_0+1}, \delta_{v,q_*+1}, \delta_{v,r_2+1}\},$$

where r_2 will be determined in (5.19). Then we conclude that if $r_1 \leqslant r \leqslant r_2$ then

$$\begin{aligned}\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r &\leqslant \|u_{0\varepsilon}\|_{L^r}^r + 2^{q-2}r(r-1)\left(\frac{\varepsilon^{m-1}}{r}\|\Delta v_{0\varepsilon}\|_{L^{r+1}}^{r+1} + \frac{1}{r+q-2}\|\Delta v_{0\varepsilon}\|_{L^{r+q-1}}^{r+q-1}\right)\\ &=: K'_{\varepsilon,r}.\end{aligned}$$

On the other hand, if $1 \leqslant r < r_1$, then we see from the Hölder inequality, the mass conservation law (3.1) and the Young inequality that

$$\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \leqslant \|u_{0\varepsilon}\|_{L^1} + \|u_\varepsilon(t)\|_{L^{r_1}(\mathbb{R}^N)}.$$

Hence we obtain the desired L^r-estimate (3.4) with $K_{\varepsilon,r} := K'^{\frac{1}{r}}_{\varepsilon,r}$ $(r_1 \leqslant r \leqslant r_2)$ and $K_{\varepsilon,r} := \|u_{0\varepsilon}\|_{L^1} + K_{\varepsilon,r_1}$ $(1 \leqslant r < r_1)$. $\square$

5. L^∞-estimates

In this section we shall show the L^∞-estimate of u_ε. The proof is based on the method of L^∞-L^r-estimates in [14, Section 4].

Lemma 5.1. *Let $N \in \mathbb{N}$, $m \geqslant 1$, $q \geqslant 2$, $\varepsilon \in (0,1)$, $T > 0$ and $0 \leqslant t_1 < t_2 \leqslant T$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(\mathrm{KS})_\varepsilon$ in Proposition 3.5. Let $\psi(t) \in C^1([t_1,t_2])$ with $0 \leqslant \psi \leqslant 1$ and $\psi(t_1) = 0$. Assume that m and q satisfy*

$$q \geqslant m + \frac{2}{N}. \tag{5.1}$$

Then for $r \geqslant q$,

$$\begin{aligned}&\psi(t_2)\|u_\varepsilon(t_2)\|_{L^{r-q+1}(\mathbb{R}^N)}^{r-q+1} + \frac{4m(r-q+1)(r-q)}{(r+m-q)^2}\int_{t_1}^{t_2}\psi(t)\left\|\nabla u_\varepsilon^{\frac{r+m-q}{2}}\right\|_{L^2(\mathbb{R}^N)}^2 dt\\ &\quad\leqslant \int_{t_1}^{t_2}\psi'(t)\|u_\varepsilon(t)\|_{L^{r-q+1}(\mathbb{R}^N)}^{r-q+1}dt + \frac{2^{q-2}(r-q+1)(r-q)}{r-1}Y_r + 2^{q-2}\varepsilon^{m-1}(r-q)Y_{r-q+2},\end{aligned} \tag{5.2}$$

where

$$Y_r := \|\Delta v_{0\varepsilon}\|_{L^r}^r + (1+C_{\langle r\rangle})\int_{t_1}^{t_2}\|u_\varepsilon(t)\|_{L^r}^r dt$$

and $C_{\langle r\rangle}$ is a constant as in Lemma 2.2.

Proof. Let $r \in (1, \infty)$. Multiplying (4.1) (see Section 4) by $\psi(t)$ and integrating it over (t_1, t_2), we see from integration by parts that

$$\frac{1}{r}\psi(t_2)\left\|u_\varepsilon(t_2)\right\|_{L^r}^r - \frac{1}{r}\int_{t_1}^{t_2} \psi'(t)\left\|u_\varepsilon(t)\right\|_{L^r(\mathbb{R}^N)}^r dt = -\int_{t_1}^{t_2} \psi(t)\boldsymbol{I}_1\,dt + \int_{t_1}^{t_2} \psi(t)\boldsymbol{I}_2\,dt,$$

where $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are those in (4.1). Since $0 \leqslant \psi \leqslant 1$, we see by an argument similar to (4.2) and (4.3) that for $r \geqslant r_1$ (see Section 4),

$$\begin{aligned}
&\psi(t_2)\left\|u_\varepsilon(t_2)\right\|_{L^r}^r - \int_{t_1}^{t_2} \psi'(t)\left\|u_\varepsilon(t)\right\|_{L^r(\mathbb{R}^N)}^r dt \\
&\leqslant -\frac{4mr(r-1)}{(r+m-1)^2}\int_{t_1}^{t_2} \psi(t)\left\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right\|_{L^2(\mathbb{R}^N)}^2 dt - \frac{4r(r-1)}{r^2}\int_{t_1}^{t_2} \psi(t)\left\|\nabla u_\varepsilon^{\frac{r}{2}}\right\|_{L^2(\mathbb{R}^N)}^2 dt \\
&\quad + \frac{2^{q-2}r(r-1)}{r+q-2}Y_{r+q-1} + \frac{2^{q-2}\varepsilon^{m-1}r(r-1)}{r}Y_{r+1}.
\end{aligned} \tag{5.3}$$

Replacing r with $r-q+1$ in (5.3), we obtain (5.2). $\square$

Lemma 5.2. *Let $N \in \mathbb{N}$, $m \geqslant 1$, $q \geqslant 2$, $\varepsilon \in (0,1)$ and $T > 0$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *in Proposition* 3.5. *Put $I = [\tau, \tau+s]$ and $I' = [\tau-\sigma, \tau+s]$ with $0 < \sigma < \tau < \tau+s < T$. Put $q_* := \frac{N}{2}(q-m)$, $h := \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{q_*}^{q_*}$ and $H := \max\{2, h\}$. Assume further that m and q satisfy* (5.1) *and δ satisfies*

$$\sigma\delta^{q-1} \leqslant 1. \tag{5.4}$$

Then for $r \geqslant q_* + q - 1$,

$$\mu_0 Z_{I,k(r-q+1)+m-1}^{\frac{1}{k}} \leqslant \left\{\frac{4}{\sigma}\delta^{-q+1} + 2^{q-2}(r-q)(2 + C_{\langle r\rangle} + C_{\langle r-q+2\rangle}) + 1\right\} Z_{I',r}, \tag{5.5}$$

where $k := 1 + \frac{2}{N}$, $\mu_0 := \min\{\nu_1, h^{1-\frac{1}{k}}, M_{v_0}^{1-\frac{1}{k}}, (s+\sigma)^{1-\frac{1}{k}}\}$, where $\nu_1 = \nu_1(m,q,N)$ is a constant,

$$Z_{I,r} := \int_I \int_{\mathbb{R}^N} u_\varepsilon^r\,dx\,dt + \frac{(s+\sigma)h}{\delta^{q_*}}\delta^r + \frac{M_{v_0}}{\|\Delta v_{0\varepsilon}\|_{L^\infty}^{q_*+q-1}}\|\Delta v_{0\varepsilon}\|_{L^\infty}^r + \frac{s+\sigma}{H^{q_*+q-1}}H^r$$

and $M_{v_0} := \|\Delta v_{0\varepsilon}\|_{L^{q_*+q-1}}^{q_*+q-1} + \|\Delta v_{0\varepsilon}\|_{L^{q_*+1}}^{q_*+1}$.

Proof. Let $r \geqslant q$. From (3.2) we can take $\tilde{t} \in I$ such that

$$\max_{t\in I}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t)\,dx = \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(\tilde{t})\,dx.$$

Put

$$Y_r' := \|\Delta v_{0\varepsilon}\|_{L^r}^r + (1 + C_{\langle r\rangle}) \int_{I'} \|u_\varepsilon(t)\|_{L^r}^r \, dt.$$

Setting

$$\tilde{\psi}(t) := \frac{t - \tau + \sigma}{\tilde{t} - \tau + \sigma}, \qquad \tilde{t_1} := \tau - \sigma, \qquad \tilde{t_2} := \tilde{t}$$

and noting that $0 \leqslant \tilde{\psi} \leqslant 1$, $\tilde{\psi}(\tilde{t_1}) = 0$, $\tilde{\psi}(\tilde{t_2}) = 1$, $0 \leqslant \tilde{\psi}'(t) = \frac{1}{\tilde{t} - \tau + \sigma} \leqslant \frac{1}{\sigma}$ and $[\tilde{t_1}, \tilde{t_2}] \subset I'$, we can substitute $\tilde{\psi}$, $\tilde{t_1}$ and $\tilde{t_2}$ into ψ, t_1 and t_2 in (5.2). Hence we have

$$\begin{aligned} &\max_{t \in I} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t) \, dx \\ &\quad \leqslant \frac{1}{\sigma} \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1} \, dx \, dt + \frac{2^{q-2}(r-q+1)(r-q)}{r-1} Y_r' + 2^{q-2} \varepsilon^{m-1} (r-q) Y_{r-q+2}'. \end{aligned} \tag{5.6}$$

Next setting

$$\hat{\psi}(t) := \begin{cases} 1, & t \in [\tau, \tau + s], \\ -\sigma^{-2}(t-\tau)^2 + 1, & t \in [\tau - \sigma, \tau], \end{cases} \qquad \hat{t_1} := \tau - \sigma, \qquad \hat{t_2} := \tau + s$$

and noting that $0 \leqslant \hat{\psi} \leqslant 1$, $\hat{\psi}(\hat{t_1}) = 0$, $\hat{\psi}(\hat{t_2}) = 1$, $0 \leqslant \hat{\psi}'(t) \leqslant \frac{2}{\sigma}$ and $I \subset [\hat{t_1}, \hat{t_2}] \subset I'$, we can substitute $\hat{\psi}$, $\hat{t_1}$ and $\hat{t_2}$ into ψ, t_1 and t_2 in (5.2). Hence we see that

$$\begin{aligned} &\frac{4m(r-q+1)(r-q)}{(r+m-q)^2} \int_I \big\|\nabla u_\varepsilon^{\frac{r+m-q}{2}}\big\|_{L^2(\mathbb{R}^N)}^2 \, dt \\ &\quad \leqslant \frac{2}{\sigma} \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1} \, dx \, dt + \frac{2^{q-2}(r-q+1)(r-q)}{r-1} Y_r' + 2^{q-2} \varepsilon^{m-1} (r-q) Y_{r-q+2}'. \end{aligned} \tag{5.7}$$

Set $q_* := \frac{N}{2}(q-m)$ and put $\nu_0 := \min\{1, \inf_{r \geqslant q_* + q - 1} \frac{4m(r-q+1)(r-q)}{(r+m-q)^2}\}$. Combining (5.6) with (5.7), we have

$$\begin{aligned} &\max_{t \in I} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t) \, dx + \nu_0 \int_I \big\|\nabla u_\varepsilon^{\frac{r+m-q}{2}}\big\|_{L^2(\mathbb{R}^N)}^2 \, dt \\ &\quad \leqslant \frac{3}{\sigma} \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1} \, dx \, dt \\ &\qquad + \frac{2^{q-1}(r-q+1)(r-q)}{r-1} \left[\|\Delta v_{0\varepsilon}\|_{L^r}^r + (1 + C_{\langle r\rangle}) \int_{I'} \|u_\varepsilon(t)\|_{L^r}^r \, dt \right] \\ &\qquad + 2^{q-1} \varepsilon^{m-1} (r-q) \left[\|\Delta v_{0\varepsilon}\|_{L^{r-q+2}}^{r-q+2} + (1 + C_{\langle r-q+2\rangle}) \int_{I'} \|u_\varepsilon(t)\|_{L^{r-q+2}}^{r-q+2} \, dt \right]. \end{aligned} \tag{5.8}$$

We consider the estimate of the right-hand side of (5.8). Set

$$\boldsymbol{E}_\delta(t) := \{x \in \mathbb{R}^N;\ u_\varepsilon(x,t) \geqslant \delta\}, \qquad q_* := \frac{N}{2}(q-m), \qquad h := \sup_{t\in[0,T]} \|u_\varepsilon(t)\|^{q_*}_{L^{q_*}(\mathbb{R}^N)}.$$

Let $r \geqslant \max\{q, q_*+q-1\} = q_*+q-1$. Noting that $|I'| = s+\sigma$, we have

$$\begin{aligned}\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}\,dx\,dt &= \left(\int_{I'}\int_{\boldsymbol{E}_\delta(t)} + \int_{I'}\int_{\mathbb{R}^N\setminus\boldsymbol{E}_\delta(t)}\right) u_\varepsilon^{r-q+1}\,dx\,dt \\ &\leqslant \delta^{-q+1}\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dx\,dt + \delta^{r-q_*-q+1}h(s+\sigma).\end{aligned} \tag{5.9}$$

Using Hölder's inequality, we see that

$$\|\Delta v_{0\varepsilon}\|^r_{L^r} + \|\Delta v_{0\varepsilon}\|^{r-q+2}_{L^{r-q+2}} \leqslant M_{v_0}\|\Delta v_{0\varepsilon}\|^{r-q_*-q+1}_{L^\infty}, \tag{5.10}$$

$$\|u_\varepsilon(t)\|^{r-q+2}_{L^{r-q+2}} \leqslant \|u_\varepsilon(t)\|^r_{L^r} + h \leqslant \|u_\varepsilon(t)\|^r_{L^r} + H^{r-q_*-q+1}, \tag{5.11}$$

where

$$M_{v_0} = \|\Delta v_{0\varepsilon}\|^{q_*+q-1}_{L^{q_*+q-1}} + \|\Delta v_{0\varepsilon}\|^{q_*+1}_{L^{q_*+1}}, \qquad H := \max\{2, h\}.$$

On the other hand, using Lemma 2.3 with $f = u_\varepsilon^{\frac{r+m-q}{2}}$, we see that

$$\begin{aligned}&\nu_0\left\{\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1}\,dx\,dt\right\}^{\frac{1}{k}} \\ &\quad\leqslant C(N)^{\frac{1}{k}}\left\{\max_{t\in I}\int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t)\,dx + \nu_0\int_I \left\|\nabla u_\varepsilon^{\frac{r+m-q}{2}}\right\|^2_{L^2(\mathbb{R}^N)}dt\right\}.\end{aligned} \tag{5.12}$$

Applying (5.12) and (5.9)–(5.11) to the left- and right-hand sides of (5.8), respectively, we have

$$\begin{aligned}&\nu_1\left\{\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1}\,dx\,dt\right\}^{\frac{1}{k}} \\ &\quad\leqslant \left[\frac{3}{\sigma}\delta^{-q+1} + 2^{q-1}(r-q)(2+C_{\langle r\rangle}+C_{\langle r-q+2\rangle})\right]\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dx\,dt \\ &\qquad+ \frac{3(s+\sigma)}{\sigma}\delta^{r-q_*-q+1}h \\ &\qquad+ 2^{q-1}(r-q)\left[M_{v_0}\|\Delta v_{0\varepsilon}\|^{r-q_*-q+1}_{L^\infty} + (1+C_{\langle r-q+2\rangle})(s+\sigma)H^{r-q_*-q+1}\right],\end{aligned} \tag{5.13}$$

where $\nu_1 = \nu_1(m,q,N) := \nu_0/C(N)^{\frac{1}{k}}$. Adding

$$\frac{s+\sigma}{\sigma}\delta^{r-q_*-q+1}h + M_{v_0}\|\Delta v_{0\varepsilon}\|^{r-q_*-q+1}_{L^\infty} + (s+\sigma)H^{r-q_*-q+1}$$

to the both sides of (5.13), we obtain

$$\begin{aligned}
&\nu_1\left\{\int_I\int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1}\,dx\,dt\right\}^{\frac{1}{k}} \\
&\quad+\frac{s+\sigma}{\sigma}\delta^{r-q_*-q+1}h+M_{v_0}\|\Delta v_{0\varepsilon}\|_{L^\infty}^{r-q_*-q+1}+(s+\sigma)H^{r-q_*-q+1} \\
&\leqslant\left[\frac{3}{\sigma}\delta^{-q+1}+2^{q-1}(r-q)(2+C_{\langle r\rangle}+C_{\langle r-q+2\rangle})\right]\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dx\,dt \\
&\quad+\frac{4}{\sigma}\delta^{-q+1}(s+\sigma)h\delta^{r-q_*} \\
&\quad+\left[2^{q-1}(r-q)+1\right]M_{v_0}\|\Delta v_{0\varepsilon}\|_{L^\infty}^{r-q_*-q+1} \\
&\quad+\left[2^{q-1}(r-q)(1+C_{\langle r-q+2\rangle})+1\right](s+\sigma)H^{r-q_*-q+1} \\
&\leqslant\left\{\frac{4}{\sigma}\delta^{-q+1}+2^{q-1}(r-q)(2+C_{\langle r\rangle}+C_{\langle r-q+2\rangle})+1\right\} \\
&\quad\times\left\{\int_{I'}\int_{\mathbb{R}^N} u_\varepsilon^r\,dx\,dt+\frac{(s+\sigma)h}{\delta^{q_*}}\delta^r+\frac{M_{v_0}}{\|\Delta v_{0\varepsilon}\|_{L^\infty}^{q_*+q-1}}\|\Delta v_{0\varepsilon}\|_{L^\infty}^r+\frac{s+\sigma}{H^{q_*+q-1}}H^r\right\}. \qquad (5.14)
\end{aligned}$$

Since $\sigma\delta^{q-1}\leqslant 1$ and

$$k(r-q+1)+m-1=k(r-q_*-q+1)+q_*+q-1,$$

it follows that

$$\begin{aligned}
\frac{(s+\sigma)h}{\sigma}\delta^{r-q_*-q+1}&=\left\{\frac{(s+\sigma)h}{\delta^{q_*}}\delta^{k(r-q+1)+m-1}\right\}^{\frac{1}{k}}\left(\frac{1}{\sigma\delta^{q-1}}\right)^{\frac{1}{k}}\left(\frac{s+\sigma}{\sigma}h\right)^{1-\frac{1}{k}} \\
&\geqslant h^{1-\frac{1}{k}}\left\{\frac{(s+\sigma)h}{\delta^{q_*}}\delta^{k(r-q+1)+m-1}\right\}^{\frac{1}{k}} \qquad (5.15)
\end{aligned}$$

and

$$M_{v_0}\|\Delta v_{0\varepsilon}\|_{L^\infty}^{r-q_*-q+1}=\left\{\frac{M_{v_0}}{\|\Delta v_{0\varepsilon}\|_{L^\infty}^{q_*+q-1}}\|\Delta v_{0\varepsilon}\|_{L^\infty}^{k(r-q+1)+m-1}\right\}^{\frac{1}{k}}M_{v_0}^{1-\frac{1}{k}}, \qquad (5.16)$$

$$(s+\sigma)H^{r-q_*-q+1}=\left\{\frac{(s+\sigma)}{H^{q_*+q-1}}H^{k(r-q+1)+m-1}\right\}^{\frac{1}{k}}(s+\sigma)^{1-\frac{1}{k}}. \qquad (5.17)$$

Applying (5.15)–(5.17) to the left-hand side of (5.14) and using the inequality $(A+B)^{\frac{1}{k}}\leqslant A^{\frac{1}{k}}+B^{\frac{1}{k}}$ $(A,B>0)$, we have

$$\mu_0\Bigg\{\int\limits_{I}\int\limits_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1}\,dx\,dt+\frac{(s+\sigma)h}{\delta^{q_*}}\delta^{k(r-q+1)+m-1}$$
$$+\frac{M_{v_0}}{\|\Delta v_{0\varepsilon}\|_{L^\infty}^{q_*+q-1}}\|\Delta v_{0\varepsilon}\|_{L^\infty}^{k(r-q+1)+m-1}+\frac{s+\sigma}{H^{q_*+q-1}}H^{k(r-q+1)+m-1}\Bigg\}^{\frac{1}{k}}$$
$$\leqslant\left\{\frac{4}{\sigma}\delta^{-q+1}+2^{q-2}(r-q)(2+C_{\langle r\rangle}+C_{\langle r-q+2\rangle})+1\right\}$$
$$\times\left\{\int\limits_{I'}\int\limits_{\mathbb{R}^N}u_\varepsilon^r\,dx\,dt+\frac{(s+\sigma)h}{\delta^{q_*}}\delta^r+\frac{M_{v_0}}{\|\Delta v_{0\varepsilon}\|_{L^\infty}^{q_*+q-1}}\|\Delta v_{0\varepsilon}\|_{L^\infty}^r+\frac{s+\sigma}{H^{q_*+q-1}}H^r\right\},$$

where $\mu_0:=\min\{\nu_1,h^{1-\frac{1}{k}},M_{v_0}^{1-\frac{1}{k}},(s+\sigma)^{1-\frac{1}{k}}\}$. Thus we obtain (5.5). $\square$

Lemma 5.3. *Let $N\geqslant 2$, $m\geqslant 1$, $q\geqslant 2$, $\varepsilon\in(0,1)$, $T>0$ and $0<\chi<\tau<\tau+s<T$. Let $(u_\varepsilon,v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *in Proposition* 3.5. *Assume that m and q satisfy* (5.1) *and δ satisfies*

$$\chi\delta^{q-1}\leqslant 1.$$

Then the following estimate holds:

$$\|u_\varepsilon\|_{L^\infty(\tau,\tau+s;L^\infty(\mathbb{R}^N))}^{r_2-(q_*+q-1)}$$
$$\leqslant[4\boldsymbol{B}]^{\frac{k}{k-1}}\big[2k^2\big]^{\frac{2k}{(k-1)^2}}\Bigg[\int\limits_{\tau-\chi}^{\tau+s}\int\limits_{\mathbb{R}^N}u_\varepsilon^{r_2}\,dx\,dt+\left(s+\frac{\chi}{2}\right)h\delta^{r_2-q_*}$$
$$+M_{v_0}\|v_{0\varepsilon}\|_{L^\infty}^{r_2-(q_*+q-1)}+\left(s+\frac{\chi}{2}\right)H^{r_2-(q_*+q-1)}\Bigg],\tag{5.18}$$

where $k=1+\frac{2}{N}$, $q_*=\frac{N}{2}(q-m)$, $r_2=r_2(m,q,N)$ and $\boldsymbol{B}=\boldsymbol{B}(M_{v_0},h,r_2,\chi,\delta,m,q,N)$ are positive constants.

Proof. Let $q_*:=\frac{N}{2}(q-m)$, $\lambda_0:=q_*+q-1$, $k:=1+\frac{2}{N}$ and let $\{\lambda_n\}_n\subset\mathbb{R}$ be the sequence satisfying the following:

$$\begin{cases}\lambda_n=(\lambda_{n-1}-q+1)k+m-1,\\ \lambda_1=r_2:=\max\{3,r_1,q_*+q-1\}\quad(r_1\text{: see Section 4}).\end{cases}\tag{5.19}$$

Then we have

$$\lambda_n=\lambda_0+(r_2-\lambda_0)k^{n-1}.\tag{5.20}$$

Since $k=1+\frac{2}{N}>1$, it follows that λ_n satisfies

$$\lambda_{n+1}>\lambda_n,\qquad r_2\leqslant\lambda_n\leqslant r_2k^{n-1}\quad\text{and}\quad\lim_{n\to\infty}\lambda_n=\infty.$$

Put $I_n:=[\tau-2^{-n+1}\chi,\tau+s]$ and take $\delta>0$ such that $\chi\delta^{q-1}\leqslant 1$. Then δ satisfies (5.4):

$$\big\{\big(\tau - 2^{-n}\chi\big) - \big(\tau - 2^{-n+1}\chi\big)\big\}\delta^{q-1} = \big(2^{-n}\chi\big)\delta^{q-1} \leqslant 1 \quad (n \geqslant 1).$$

Therefore we can put $r = \lambda_n$, $I = I_{n+1}$ and $I' = I_n$ in (5.5). Setting

$$\boldsymbol{J}_n := \int_{I_n}\int_{\mathbb{R}^N} u_\varepsilon^{\lambda_n}\,dx\,dt + \frac{(s+2^{-n}\chi)h}{\delta^{q_*}}\delta^{\lambda_n} + \frac{M_{v_0}}{\|\Delta v_{0\varepsilon}\|_{L^\infty}^{\lambda_0}}\|\Delta v_{0\varepsilon}\|_{L^\infty}^{\lambda_n} + \frac{s+2^{-n}\chi}{H^{\lambda_0}}H^{\lambda_n},$$

we see from (5.5) that

$$\mu_0 \boldsymbol{J}_{n+1}{}^{\frac{1}{k}} \leqslant \left\{\frac{4}{2^{-n}\chi\delta^{q-1}} + 2^{q-2}(\lambda_n - q)(2 + C_{\langle\lambda_n\rangle} + C_{\langle\lambda_n - q+2\rangle}) + 1\right\}\boldsymbol{J}_n. \tag{5.21}$$

Now we consider the estimates of the coefficients in (5.21). First, from the definition of μ_0 in Lemma 5.2, there exists the number n_0 such that

$$\mu_0 = \begin{cases} (2^{-n}\chi)^{1-\frac{1}{k}} & (n > n_0), \\ \min\{\nu_1, h^{1-\frac{1}{k}}, M_{v_0}^{1-\frac{1}{k}}\} =: \mu_{n_0} & (1 \leqslant n \leqslant n_0). \end{cases} \tag{5.22}$$

Next, noting that $2^{-n}\chi\delta^{q-1} \leqslant 1$, we see that

$$\begin{aligned} &\frac{4}{2^{-n}\chi\delta^{q-1}} + 2^{q-2}(\lambda_n - q)(2 + C_{\langle\lambda_n\rangle} + C_{\langle\lambda_n-q+2\rangle}) + 1 \\ &\quad\leqslant \frac{1}{2^{-n}\chi\delta^{q-1}}\big\{4 + 2^{q-2}(\lambda_n - q)(2 + C_{\langle\lambda_n\rangle} + C_{\langle\lambda_n-q+2\rangle}) + 1\big\} \\ &\quad\leqslant \frac{2^{q-1}}{2^{-n}\chi\delta^{q-1}}\lambda_n(2 + C_{\langle\lambda_n\rangle} + C_{\langle\lambda_n-q+2\rangle}). \end{aligned} \tag{5.23}$$

Finally, it follows from the representation (2.5) of $C_{\langle r\rangle}$ that for $r \geqslant 3$,

$$C_{\langle r\rangle} = A_0\left(A_1 r + A_2\frac{r}{r-2}\right)^{\frac{2(r-1)}{r}}\left(\frac{5^N r}{2}\right)^{\frac{r-2}{2r}} \leqslant A_3 r^{\frac{5}{2}}, \tag{5.24}$$

where $A_3 = A_3(A_1, A_2, N) = A_3(N)$ is a positive constant. Since $\lambda_n \leqslant r_2 k^{n-1}$, we see from (5.21)–(5.24) that for $n > n_0$,

$$\begin{aligned} \boldsymbol{J}_{n+1}{}^{\frac{1}{k}} &\leqslant \big(2^n\chi^{-1}\big)^{1-\frac{1}{k}}\left[\frac{2^{q-1}}{2^{-n}\chi\delta^{q-1}}r_2k^{n-1}\big(2 + A_3\big(r_2k^{n-1}\big)^{\frac{5}{2}} + A_3\big(r_2k^{n-1} - q + 2\big)^{\frac{5}{2}}\big)\right]\boldsymbol{J}_n \\ &\leqslant \big(2^n\chi^{-1}\big)^{1-\frac{1}{k}}\left[\frac{2^{q-1}r_2}{\chi\delta^{q-1}}2^n k^{n-1}\big(4A_3\big(r_2k^{n-1}\big)^{\frac{5}{2}}\big)\right]\boldsymbol{J}_n \\ &\leqslant B\big[2^{n(2-\frac{1}{k})}k^{\frac{7}{2}(n-1)}\big]\boldsymbol{J}_n \\ &\leqslant B\big[2^{2n}k^{4(n-1)}\big]\boldsymbol{J}_n, \end{aligned} \tag{5.25}$$

where

$$B := 2^{q+1}A_3 r_2^{\frac{7}{2}}\big(\chi^{-1}\big)^{2-\frac{1}{k}}\delta^{-q+1}.$$

Since the same argument as in (5.25) is carried out for $1 \leqslant n \leqslant n_0$, we have

$$\boldsymbol{J}_{n+1}{}^{\frac{1}{k}} \leqslant \boldsymbol{B}\big[2^{2n}k^{4(n-1)}\big]\boldsymbol{J}_n \quad (n \geqslant 1),$$

where

$$\boldsymbol{B} := \max\big\{B, \mu_{n_0}^{-1}2^{q+1}A_3 r_2^{\frac{7}{2}}\big(\chi\delta^{q-1}\big)^{-1}\big\}.$$

Therefore we obtain

$$\begin{aligned}\{\boldsymbol{J}_{n+1}\}^{\frac{1}{k^n}} &\leqslant \big[\boldsymbol{B}2^{2n}k^{4(n-1)}\big]^{\frac{1}{k^{n-1}}}\big[\boldsymbol{B}2^{2(n-1)}k^{4(n-2)}\big]^{\frac{1}{k^{n-2}}} \times \cdots \times \big[\boldsymbol{B}2^2\big]\boldsymbol{J}_1\\ &= [4\boldsymbol{B}]^{\frac{1}{k^{n-1}}+\frac{1}{k^{n-2}}+\cdots+1}\big[2k^2\big]^{\frac{2(n-1)}{k^{n-1}}+\frac{2(n-2)}{k^{n-2}}+\cdots+\frac{2}{k}}\boldsymbol{J}_1. \end{aligned} \tag{5.26}$$

From the definition of $\boldsymbol{J}_n$ and (5.20) we see that

$$\begin{aligned}\liminf_{n\to\infty}\{\boldsymbol{J}_{n+1}\}^{\frac{1}{k^n}} &\geqslant \liminf_{n\to\infty}\left\{\int_{\tau-2^{-n}\chi}^{\tau+s}\int_{\mathbb{R}^N} u_\varepsilon^{\lambda_{n+1}}\,dx\,dt\right\}^{\frac{r_2-\lambda_0}{\lambda_{n+1}-\lambda_0}}\\ &\geqslant \liminf_{n\to\infty}\|u_\varepsilon\|_{L^{\lambda_{n+1}}(\tau,\tau+s;L^{\lambda_{n+1}}(\mathbb{R}^N))}^{\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_0}\cdot r_2-\lambda_0}\\ &= \|u_\varepsilon\|_{L^\infty(\tau,\tau+s;L^\infty(\mathbb{R}^N))}^{r_2-\lambda_0}.\end{aligned}$$

Therefore it follows from (5.26) that

$$\begin{aligned}&\|u_\varepsilon\|_{L^\infty(\tau,\tau+s;L^\infty(\mathbb{R}^N))}^{r_2-\lambda_0}\\ &\quad\leqslant \liminf_{n\to\infty}\{\boldsymbol{J}_{n+1}\}^{\frac{1}{k^n}}\\ &\quad\leqslant \limsup_{n\to\infty}[4\boldsymbol{B}]^{\frac{1}{k^{n-1}}+\frac{1}{k^{n-2}}+\cdots+1}\big[2k^2\big]^{\frac{2(n-1)}{k^{n-1}}+\frac{2(n-2)}{k^{n-2}}+\cdots+\frac{2}{k}}\boldsymbol{J}_1\\ &\quad= [4\boldsymbol{B}]^{\frac{k}{k-1}}\big[2k^2\big]^{\frac{2k}{(k-1)^2}}\\ &\qquad\times\left[\int_{\tau-\chi}^{\tau+s}\int_{\mathbb{R}^N} u_\varepsilon^{r_2}\,dx\,dt + \left(s+\frac{\chi}{2}\right)h\delta^{r_2-q_*} + M_{v_0}\|v_{0\varepsilon}\|_{L^\infty}^{r_2-\lambda_0} + \left(s+\frac{\chi}{2}\right)H^{r_2-\lambda_0}\right].\end{aligned}$$

Hence we obtain (5.18). $\square$

Proof of Proposition 3.3. Let $0 < \tau < \tau + s < T$. Taking χ and δ such that $\chi = \delta^{-(q-1)} = \frac{\tau}{2}$ in (5.18), we see from (3.4) that

$$\begin{aligned}&\|u_\varepsilon\|_{L^\infty(\tau,\tau+s;L^\infty(\mathbb{R}^N))}^{r_2-(q_*+q-1)}\\ &\quad\leqslant [4\boldsymbol{B}_*]^{\frac{k}{k-1}}\big[2k^2\big]^{\frac{2k}{(k-1)^2}}\left[\left(s+\frac{\tau}{2}\right)K_{\varepsilon,r_2}^{r_2} + \left(s+\frac{\tau}{4}\right)\left(\frac{\tau}{2}\right)^{\frac{r_2-q_*}{-(q-1)}}h\right.\end{aligned}$$

$$
\begin{aligned}
&\quad + M_{v_0}\|v_{0\varepsilon}\|_{L^\infty}^{r_2-(q_*+q-1)} + \left(s+\frac{\tau}{4}\right)H^{r_2-(q_*+q-1)}\Bigg] \\
&\leqslant [4\boldsymbol{B_*}]^{\frac{k}{k-1}}\left[2k^2\right]^{\frac{2k}{(k-1)^2}}\left[TK_{\varepsilon,r_2}^{r_2} + T^{1+\frac{r_2-q_*}{-(q-1)}}h + M_{v_0}\|v_{0\varepsilon}\|_{L^\infty}^{r_2-(q_*+q-1)} + TH^{r_2-(q_*+q-1)}\right] \\
&=: K_{\varepsilon,\infty}, \qquad (5.27)
\end{aligned}
$$

where $\boldsymbol{B_*} := 2^{q+1}A_3 r_2^{\frac{7}{2}}\max\{(\frac{2}{\tau})^{1-\frac{1}{k}}, \mu_{n_0}^{-1}\}$. Thus we obtain (3.5). □

Remark 5.1. From the definition of $u_{0\varepsilon}$ and $v_{0\varepsilon}$: $u_{0\varepsilon} := (u_0 * \rho_\varepsilon)\zeta_\varepsilon$ and $v_{0\varepsilon} := (v_0 * \rho_\varepsilon)\zeta_\varepsilon$, it follows that

$$
\|u_{0\varepsilon}\|_{L^r} \leqslant \|u_0\|_{L^r} \quad \big(\forall r \in [1,\infty]\big), \qquad \|\Delta v_{0\varepsilon}\|_{L^r} \leqslant \|\Delta v_0\|_{L^r} + 1 \quad \left(\forall r \in \left[\frac{N}{2}(q-m)+1, \infty\right]\right).
$$

Hence the definitions of $K_{\varepsilon,r}$ and $K_{\varepsilon,\infty}$ (see Section 4 and (5.27), respectively) imply that

$$
\begin{gathered}
K_{\varepsilon,r} < K_r, \quad 1 \leqslant r \leqslant r_2, \\
K_{\varepsilon,\infty} < K_\infty,
\end{gathered}
$$

where K_r and K_∞ are constants independent of ε; note that $\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \leqslant K_r := K_1 + K_\infty$ ($r_2 \leqslant r < \infty$).

6. Proof of Theorem 1.1

In this section we discuss the convergence of approximate solutions. The following three lemmas are obtained by similar arguments to those in [6, Lemmas 5.1, 5.2 and 5.3].

Lemma 6.1. *Let $N \geqslant 2$, $m \geqslant 1$, $q \geqslant 2$, $\varepsilon \in (0,1)$ and $T > 0$. Let (u_0, v_0) satisfy* (1.1) *and* (1.2). *Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *in Proposition* 3.5. *Assume that*

$$
q \geqslant m + \frac{2}{N} \qquad (6.1)
$$

and (u_0, v_0) satisfies the smallness conditions as in Theorem 1.1. *Then the following estimate holds*:

$$
\begin{aligned}
&\frac{4m}{(m+1)^2}\left\|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right\|_{L^2(0,T;L^2(\mathbb{R}^N))}^2 \\
&\quad \leqslant \frac{1}{2}\|u_0\|_{L^2(\mathbb{R}^N)}^2 + M_0T\left(\|\Delta v_0\|_{L^{q_*}(\mathbb{R}^N)} + 1 + C_{q_*,N}K_{q_*}\right),
\end{aligned}
$$

where $q_* = \frac{N}{2}(q-m)$ and $M_0 = M_0(q, K_{\frac{q(q_*+1)}{q_*}}, K_{\frac{2(q_*+1)}{q_*}}) > 0$ is a constant and K_r is given in Remark 5.1.

Lemma 6.2. *Let $N \geqslant 2$, $m \geqslant 1$, $q \geqslant 2$, $\varepsilon \in (0,1)$ and $T > 0$. Let (u_0, v_0) satisfy* (1.1) *and* (1.2). *Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to* $(\mathrm{KS})_\varepsilon$ *in Proposition* 3.5. *Assume that m and q satisfy* (6.1) *and (u_0, v_0) satisfies the smallness conditions as in Theorem* 1.1. *Then the following estimate holds*:

$$
\left\|\sqrt{t}\frac{\partial}{\partial t}u_\varepsilon^m\right\|_{L^2(0,T;L^2(\mathbb{R}^N))}^2 + \sup_{0<t<T}\left\|\sqrt{t}\nabla u_\varepsilon^m(t)\right\|_{L^2(\mathbb{R}^N)}^2 \leqslant M_0',
$$

where $M_0' > 0$ is a constant independent of ε.

Lemma 6.3. *Let $N \geqslant 2$, $m \geqslant 1$, $q \geqslant 2$, $\varepsilon \in (0,1)$, $T > 0$. Let (u_0, v_0) satisfy* (1.1), (1.2). *Let $(u_\varepsilon, v_\varepsilon)$ be a unique non-negative global solution to* $(\mathrm{KS})_\varepsilon$ *in Proposition* 3.5. *Assume that m and q satisfy* (6.1) *and (u_0, v_0) satisfies the smallness conditions as in Theorem* 1.1. *Then there exist subsequences $\{u_{\varepsilon_n}\}$, $\{v_{\varepsilon_n}\}$ and non-negative functions u, v such that $u \in L^\infty(0,T;L^p(\mathbb{R}^N))$ $(\forall p \in [1,\infty])$, $u^{\frac{m+1}{2}} \in L^2(0,T;H^1(\mathbb{R}^N))$, $v \in L^\infty(0,T;W^{1,p}(\mathbb{R}^N))$ $(\forall p \in [2,\infty])$ and*

$$u_{\varepsilon_n} \to u \quad \text{weakly* in } L^\infty\big(0,T;L^p\big(\mathbb{R}^N\big)\big) \quad \big(\forall p \in [1,\infty]\big), \tag{6.2}$$

$$u_{\varepsilon_n} \to u \quad \text{strongly in } C\big([\delta,T];L^p_{\mathrm{loc}}\big(\mathbb{R}^N\big)\big) \quad \big(\forall \delta > 0,\ \forall p \in [1,\infty)\big), \tag{6.3}$$

$$\nabla(u_{\varepsilon_n} + \varepsilon_n)^{\frac{m+1}{2}} \to \nabla u^{\frac{m+1}{2}} \quad \text{weakly in } L^2\big(0,T;L^2\big(\mathbb{R}^N\big)\big), \tag{6.4}$$

$$v_{\varepsilon_n} \to v \quad \text{weakly* in } L^\infty\big(0,T;L^p\big(\mathbb{R}^N\big)\big) \quad \big(\forall p \in [2,\infty]\big), \tag{6.5}$$

$$\nabla v_{\varepsilon_n} \to \nabla v \quad \text{weakly* in } L^\infty\big(0,T;L^p\big(\mathbb{R}^N\big)\big) \quad \big(\forall p \in [2,\infty]\big). \tag{6.6}$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $T > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^N \times [0,T))$. Multiplying $(1)_{\varepsilon_n}$ and $(2)_{\varepsilon_n}$ (see Section 3) by φ and integrating those on $\mathbb{R}^N \times (0,T)$, we see that

$$\int_0^T \int_{\mathbb{R}^N} \Big(\nabla(u_{\varepsilon_n} + \varepsilon_n)^m \cdot \nabla\varphi - \big(u_{\varepsilon_n} + \varepsilon_n^{\frac{m-1}{q-2}}\big)^{q-2} u_{\varepsilon_n} \nabla v_{\varepsilon_n} \cdot \nabla\varphi - u_{\varepsilon_n}\varphi_t\Big)\,dx\,dt$$
$$= \int_{\mathbb{R}^N} u_{0_{\varepsilon_n}}(x)\varphi(x,0)\,dx,$$

$$\int_0^T \int_{\mathbb{R}^N} (\nabla v_{\varepsilon_n} \cdot \nabla\varphi + v_{\varepsilon_n}\varphi - u_{\varepsilon_n}\varphi - v_{\varepsilon_n}\varphi_t)\,dx\,dt = \int_{\mathbb{R}^N} v_{0_{\varepsilon_n}}(x)\varphi(x,0)\,dx.$$

First we deal with $(1)_{\varepsilon_n}$. Using (6.3), (6.4) and the estimate $\|u_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^N))} \leqslant K_\infty$, we see that $u^m \in L^2(0,T;H^1(\mathbb{R}^N))$ and

$$\int_0^T \int_{\mathbb{R}^N} \nabla(u_{\varepsilon_n} + \varepsilon_n)^m \cdot \nabla\varphi\,dx\,dt \to \int_0^T \int_{\mathbb{R}^N} \nabla u^m \cdot \nabla\varphi\,dx\,dt.$$

On the other hand, (6.3) implies that

$$\big(u_{\varepsilon_n} + \varepsilon_n^{\frac{m-1}{q-2}}\big)^{q-2} u_{\varepsilon_n} \nabla\varphi \to u^{q-1}\nabla\varphi \qquad \text{a.a. } x \in \mathbb{R}^N,\ t \in (0,T).$$

The estimate $\|u_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^N))} \leqslant K_\infty$ gives

$$\big(u_{\varepsilon_n} + \varepsilon_n^{\frac{m-1}{q-2}}\big)^{q-2} u_{\varepsilon_n} |\nabla\varphi| \leqslant (K_\infty + 1)^{q-2} K_\infty |\nabla\varphi| \in L^1\big(0,T;L^2\big(\mathbb{R}^N\big)\big).$$

Hence we see from the Lebesgue dominated convergence theorem that

$$\big(u_{\varepsilon_n} + \varepsilon_n^{\frac{m-1}{q-2}}\big)^{q-2} u_{\varepsilon_n} \nabla\varphi \to u^{q-1}\nabla\varphi \quad \text{strongly in } L^1\big(0,T;L^2\big(\mathbb{R}^N\big)\big).$$

This together with (6.6) shows that

$$\int_0^T \int_{\mathbb{R}^N} \left(u_{\varepsilon_n} + \varepsilon_n^{\frac{m-1}{q-2}}\right)^{q-2} u_{\varepsilon_n} \nabla v_{\varepsilon_n} \cdot \nabla \varphi \, dx\, dt \to \int_0^T \int_{\mathbb{R}^N} u^{q-1} \nabla v \cdot \nabla \varphi \, dx\, dt.$$

Using (6.2) or (6.3) and recalling the definition of $u_{0\varepsilon}$: $u_{0\varepsilon} = (u_0 * \rho_\varepsilon)\zeta_\varepsilon$, we see that

$$\int_0^T \int_{\mathbb{R}^N} u_{\varepsilon_n} \varphi_t \, dx\, dt \to \int_0^T \int_{\mathbb{R}^N} u \varphi_t \, dx\, dt,$$

$$\int_{\mathbb{R}^N} u_{0_{\varepsilon_n}}(x) \varphi(x,0)\, dx \to \int_{\mathbb{R}^N} u_0(x) \varphi(x,0)\, dx.$$

Next, it follows from (6.5) and (6.6) that

$$\int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \varphi + v\varphi - u\varphi - v\varphi_t)\, dx\, dt = \int_{\mathbb{R}^N} v_0(x) \varphi(x,0)\, dx.$$

Therefore the above convergences imply that (u, v) is the non-negative global solution to (KS). Finally we prove (1.3) and (1.4). It follows from (6.2) and Remark 5.1 that

$$\|u\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} \leqslant \liminf_{n\to\infty} \|u_{\varepsilon_n}\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} \leqslant K_r \quad \big(\forall r \in [1, \infty]\big). \tag{6.7}$$

Applying (2.1) with $q = p = r$ and noting that $\|v_{0\varepsilon}\|_{L^r} \leqslant \|v_0\|_{L^r}$ ($\forall r \in [\frac{N}{2}(q-m)+1, \infty]$), we see from Remark 5.1 that for $t \in (0, T)$,

$$\big\|v_\varepsilon(t)\big\|_{L^r(\mathbb{R}^N)} \leqslant \|v_0\|_{L^r(\mathbb{R}^N)} + C_0 K_r \quad \left(\forall r \in \left[\frac{N}{2}(q-m)+1, \infty\right]\right). \tag{6.8}$$

Combining (6.5) with (6.8), we have

$$\|v\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} \leqslant \|v_0\|_{L^r(\mathbb{R}^N)} + C_0 K_r \quad \left(\forall r \in \left[\frac{N}{2}(q-m)+1, \infty\right]\right). \tag{6.9}$$

Therefore (1.3) follows from (6.7) and (6.9). Moreover, applying (2.4) with $p = r$ implies that for each $r \in [\frac{N}{2}+1, \infty)$,

$$\begin{aligned}\|\Delta v_\varepsilon\|_{L^r(0,T;L^r(\mathbb{R}^N))} &\leqslant \|\Delta v_{0\varepsilon}\|_{L^r(\mathbb{R}^N)} \big(1 - e^{-rt}\big)^{\frac{1}{r}} + C_{\langle r\rangle} \|u_\varepsilon\|_{L^r(0,T;L^r(\mathbb{R}^N))} \\ &\leqslant \|\Delta v_{0\varepsilon}\|_{L^r(\mathbb{R}^N)} + C_{\langle r\rangle} K_{\varepsilon,r} T^{\frac{1}{r}}.\end{aligned}$$

From Remark 5.1, we have

$$\|\Delta v_\varepsilon\|_{L^r(0,T;L^r(\mathbb{R}^N))} \leqslant \|\Delta v_0\|_{L^r(\mathbb{R}^N)} + 1 + C_{\langle r\rangle} K_r T^{\frac{1}{r}}.$$

This inequality and (1.3) yield (1.4). This completes the proof of Theorem 1.1. $\square$

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